1 Definitions

We'll start by mathing up the beat data structure and functions described in the post.

We'll define a **beat** q as a triple (t, d, b), where:

- $t \in \mathbb{R}$ is the metronome starting time (`startTime`),
- $d \in \mathbb{R}^+$ is the duration of a thirty-second note in the beat's tempo (`thirtysecondDuration`), and
- $b \in \mathbb{R}$ is the number of thirty-second notes to count from the start time (`thirtyseconds`).

We'll also define the set of beats $\mathbb{B} = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, so I don't have to paste that Cartesian product six more times through this thing.

Next, we'll define the **timestamp function** $T : \mathbb{B} \to \mathbb{R}$:

$$T(q) = t + db \tag{1}$$

Last, we'll define the **tempo conversion function** $C : \mathbb{B} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{B}$, which takes a beat ($q$), pivot time ($t_p$), and new thirty-second note duration (d') and gives a new beat. Using the proposed definition in the post:

$$C(q, t_p, d') = (t_p, d', \frac{T(q) - t_p}{d}) \tag{2}$$

2 Proportionality

Now, what we're trying to show is that the conversion function C meets the second property in the post, which is that given two beats and a pivot time between them, if we convert both beats to a different tempo, then the pivot time is proportionally the same distance between the two endpoints of the new interval that it is in the old interval.

But oops, we have to define one more thing here: what does "proportionality" mean? Well, if the pivot time is equal to the left endpoint then that's 0% (0.0) of the way through the interval, and if it's equal to the right endpoint then that's 100% (1.0), and if it's half the distance between the two then that's 50% (0.5), and so on. What we're doing is finding how far we are from the left endpoint (the distance between $T(A)$ and t_p) and dividing that by the length of the interval, which sounds like a **proportionality function** $P : \mathbb{B} \times \mathbb{B} \times \mathbb{R} \to \mathbb{R}$, where:

$$P(A, B, t_p) = \frac{\text{distance}}{\text{length}} = \frac{t_p - T(A)}{T(B) - T(A)} \tag{3}$$

Good! This is a thing we can do math with.

3 But we can't quite start yet because first we need to prove that the converted interval is in fact a proper interval

So, we're given $A, B \in \mathbb{B}$ with the same tempo:

$$\begin{aligned} A &= (t_A, d, b_A) \\ B &= (t_B, d, b_B) \end{aligned} \tag{4}$$

Let's throw in the condition that $T(A) < T(B)$, so that the interval does in fact span time and we don't have any unfortunate incidents when we divide by $T(B) - T(A)$ later.

We're also given a pivot time $t_p \in \mathbb{R}$ such that $T(A) \leq t_p \leq T(B)$, and a new thirty-second note duration $d' \in \mathbb{R}^+$. So we make the converted interval like so:

$$\begin{aligned} A' &= C(A, t_p, d') = (t_p, d', \frac{T(A) - t_p}{d}) \\ B' &= C(B, t_p, d') = (t_p, d', \frac{T(B) - t_p}{d}) \end{aligned} \tag{5}$$

But we need to show that this is also a proper interval with t_p inside it before we move on. So first, let's show that $T(A') < T(B')$:

$$\begin{aligned} T(A') - T(B') &= t_p + d' \frac{T(A) - t_p}{d} - t_p - d' \frac{T(B) - t_p}{d} \\ &= \frac{d'}{d}(T(A) - t_p - T(B) + t_p) \\ &= \frac{d'}{d}(T(A) - T(B)) \\ &< 0 \text{ because } T(A) < T(B) \end{aligned}$$

And now we show that $T(A') \leq t_p \leq T(B')$. Let's start with $T(A') \leq t_p$:

$$\begin{aligned} T(A') &= t_p + d' \frac{T(A) - t_p}{d} \\ &= t_p + \frac{d'}{d}(T(A) - t_p) \\ &\leq t_p \text{ because } T(A) - t_p \leq 0 \end{aligned}$$

The proof that $T(B') \geq t_p$ works in basically the same way, except that $T(B) - t_p \geq 0$ proves the inequality in the other direction.

4 The part where we prove the thing

Jesus, finally. To recap, from equations 4 and 5 we have:

$$\begin{aligned}
A &= (t_A, d, b_A) \\
B &= (t_B, d, b_B) \\
A' &= (t_p, d', \frac{T(A) - t_p}{d}) \\
B' &= (t_p, d', \frac{T(B) - t_p}{d})
\end{aligned}$$

And what we need to show is that $P(A, B, t_p) = P(A', B', t_p)$.

First, we need a goal:

$$P(A, B, t_p) = \frac{t_p - T(A)}{T(B) - T(A)}$$

And now we'll do some algebra with the second interval:

$$\begin{aligned}
P(A', B', t_p) &= \frac{t_p - T(A')}{T(B') - T(A')} \\
&= \frac{t_p - t_p - d' \frac{T(A)-t_p}{d}}{t_p + d' \frac{T(B)-t_p}{d} - t_p - d' \frac{T(A)-t_p}{d}} \\
&= \frac{-\frac{d'}{d}(T(A) - t_p)}{\frac{d'}{d}(T(B) - t_p - T(A) + t_p)} \\
&= \frac{t_p - T(A)}{T(B) - T(A)} \\
&= P(A, B, t_p)
\end{aligned}$$

So it checks out! Thanks, math, for explaining something that only took seventy words in the actual post. You're the best.